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Journal of Computational Physics 227 (2007) 431–439

JOURNAL OF COMPUTATIONAL PHYSICS

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An inverse scattering problem from an impedance obstacle

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Received 18 April 2007; received in revised form 25 July 2007; accepted 29 July 2007
Available online 10 August 2007

Abstract

In this paper we consider an inverse scattering problem from an obstacle with impedance boundary condition. Our aim is to recover the unknown scatterer from the far field pattern iteratively assuming the impedance function. Our method, while remaining in the framework of Newton's method, based on a system of two nonlinear integral equations which is equivalent to the original inverse problem, avoids the need of calculating a direct problem at each iteration. Because of the ill-posedness of this problem, regularization method for example, Tikhonov regularization, is incorporated in our solution scheme. Several numerical examples with only one incident wave are given at the end of the paper to show the feasibility of our method.

Keywords: Inverse scattering; Impedance boundary condition; Helmholtz equation

1. Introduction

The inverse scattering problem which aim is the recovering of the geometry of the obstacles or the investigating of the physical properties of the scatterers has attracted more and more attentions in the past decades not only because of its pure mathematical interest but also of its applicability in the real world. The inverse scattering problem that we are considering in this paper is to find an impedance scatterer in the domain concerned. The impedance boundary conditions can be used to model practical problems like surface coating which has its application in detection of buried objects, antenna design or the analysis of the earth surface (see [14]), for example. In [1] a direct method was given to determine the impedance function for an arbitrary fixed boundary. In this paper, on the other hand, we try to recover the unknown scatterer for an arbitrary given impedance function.

In the paper from Kress and Rundell [8], an inverse impedance problem for recovering both the scatterer and the impedance are considered. Based on the regularized Newton's method, their method needed to solve a direct problem at each iteration step. Although the problem was splitted into two smaller parts, it is still time consuming. Recently, Kress and Rundell [9] proposed a new method which is also a Newton-like method, but it avoids the need of a forward solver at every iteration. Based on the gap functional, they introduced a system

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2007.07.030

of nonlinear integral equations which is equivalent to the original inverse problem by putting an auxiliary curve around the obstacle. Thus to solve the inverse problem one has only to solve the equivalent system. This method has been extended to other boundary conditions and cracks, see [4,5]. Using the far field pattern, a different method was proposed in [10] to solve a Neumann problem which also leads to a system of two nonlinear integral equations with the advantage of no need of an auxiliary curve. In this paper we'd like to extend this method to the more general case of an impedance problem.

At this place we want to mention that there are several methods for treating impedance problem. The treatment in [13] recovered both the impedance and the scatterer, however, far field data at many frequencies were needed. Another method developed by Colton and Kirsch [2] reconstructed the unknown scatterer without knowledge of the type of the boundary condition. The drawback of their method lies in the fact that all measurements from all incident directions are needed. More recently, the hybrid method described in [12] which reformulates the inverse problem as an optimization problem, recovers successfully both the impedance and the obstacle.

The plan of the paper is as follows. For the sake of completeness and also the introduction of notations, in Section 2 we will briefly summarize the main results of the direct problem. In Section 3 we will formulate the inverse scattering problem and derive an equivalent system of two nonlinear integral equations whose solution is the solution of the original inverse problem. The numerical scheme will be then given in Section 4 which is followed by some numerical examples in the final section.

2. Direct scattering problem

Problem 1 (*Direct impedance scattering problem*). Given an incident plane wave $u^{\mathrm{i}}(x,d):=\mathrm{e}^{\mathrm{i}k\langle x,d\rangle}$ with a fixed wave number $k\in\mathbb{R}$ and an incident direction $d\in\Omega:=\{x\in\mathbb{R}^2||x|=1\}$, the unit circle in $\mathbb{R}^2$, find the scattered field u^{s} such that the total field $u:=u^{\mathrm{i}}+u^{\mathrm{s}}$ satisfies the Helmholtz equation

$$\Delta u+k^2u=0 \tag{1}$$

in the exterior of the scatterer D and the impedance boundary condition

$$\frac{\partial u}{\partial \nu}+\mathrm{i}k\lambda u=0\quad \text{on } \partial D \tag{2}$$

for an impedance function $\lambda\in C^{0,\zeta}(\partial D)$ with $\mathrm{Im}(\bar{k}\lambda)\geqslant 0$ on ∂D. The scattered field u^{s} itself fulfills the Sommerfeld radiation condition

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^{\mathrm{s}}}{\partial \nu}-\mathrm{i}ku^{\mathrm{s}}\right)=0,\quad r=|x| \tag{3}$$

uniformly in all directions.

Using integral equation method, the direct problem can be uniquely solved. Firstly, we define a solution ansatz by a combination of a double layer potential and a single layer potential in terms of a yet to be determined density function $\varphi\in C^{1,\zeta}(\partial D)$

$$u^{\mathrm{s}}(x):=\int_{\partial D}\Phi(x,y)\varphi(y)\,\mathrm{d}s(y)+\mathrm{i}\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\,\mathrm{d}s(y) \tag{4}$$

where

$$\Phi(x,y):=\frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|),\quad x\neq y$$

denotes the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ in terms of the Hankel function $H_0^{(1)}$ of the first kind and of order zero. By substituting this ansatz into the impedance boundary condition (2) and using the jump relations of the layer potentials we see that the density φ is required to be a solution of the following equation:

$$\{\mathrm{i}T+K'-I+\mathrm{i}k\lambda(S+\mathrm{i}K+\mathrm{i}I)\}\varphi=f \tag{5}$$

where

$$f = -2\left(\frac{\partial u^{\mathrm{i}}}{\partial \nu} + \mathrm{i}k\lambda u^{\mathrm{i}}\right)$$

and the operators S, K, K', T are defined by

$$S\varphi(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)\,\mathrm{d}s(y) \tag{6}$$

$$K\varphi(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\,\mathrm{d}s(y) \tag{7}$$

$$K'\varphi(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)\,\mathrm{d}s(y) \tag{8}$$

$$T\varphi(x) := 2\frac{\partial}{\partial \nu(x)}\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\,\mathrm{d}s(y) \tag{9}$$

The unique solvability of the boundary integral equation (5) can be derived from the Riesz theory. We refer to the monograph [3]. For the unique solvability of the direct impedance problem, we have the following theorem.

Theorem 1. *The direct impedance Problem 1 is uniquely solvable and the solution is given by the solution ansatz (4) with the density function $\varphi \in C^{1,\zeta}(\partial D)$ which solves (5).*

At this place we want to point out that in the scattering theory, the behavior of the scattered field at a large distance is of particular interest at least for the data availability. The far field pattern describes the behavior of the scattered wave at the infinity

$$u^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|}\right)\right\} \quad |x| \to \infty$$

uniformly in all directions $\hat{x} \in \Omega$. The one-to-one correspondence between the radiating waves and their far field patterns is established by the Rellich's lemma. Making use of the asympototics of the Bessel functions, the far field pattern of the scattered field for the impedance problem is easily found to be

$$u_\infty(\hat{x}) = \xi_1 \int_\Gamma \mathrm{e}^{-\mathrm{i}k\langle \hat{x},y\rangle}\varphi(y)\,\mathrm{d}s(y) + \mathrm{i}\xi_2\int_\Gamma \langle \nu(y),\hat{x}\rangle \mathrm{e}^{-\mathrm{i}k\langle \hat{x},y\rangle}\varphi(y)\,\mathrm{d}s(y) \tag{10}$$

with the constants $\xi_1 = \frac{1+\mathrm{i}}{4\sqrt{k\pi}}$, $\xi_2 = \frac{1-\mathrm{i}}{4}\sqrt{\frac{k}{\pi}}$ and the density function φ given by Theorem 1.

To reduce the hypersingularity of the operator T, we use the Maue's identity to split it into two milder parts (for a proof of this splitting, see for example Theorem 7.29 in [7])

$$T\varphi = \frac{\partial}{\partial \vartheta} S \frac{\partial \varphi}{\partial \vartheta} + k^2\langle \nu, S\varphi\nu\rangle,$$

where ϑ is the unit tangent vector. The integral equation (5) now becomes

$$\mathrm{i}\frac{\partial}{\partial \vartheta} S \frac{\partial \varphi}{\partial \vartheta} + \mathrm{i}k^2\langle \nu, S\varphi\nu\rangle + K'\varphi - \varphi + \mathrm{i}k\lambda(S + \mathrm{i}K + \mathrm{i}I)\varphi = f \tag{11}$$

This equation can be numerically solved by the Nyström method as in [7], for example.

3. Inverse scattering problem

After introducing the notations in the last section, we consider the following inverse problem:

Problem 2 (*Inverse impedance problem*). Determine the scatterer D if the far field pattern u_∞ is known for one incident plane wave assuming a prescribed impedance function λ.

For our inverse problem it is noted that there is no uniqueness result available. Although it is widely believed that the inverse problem for only one incident wave is uniquely solvable, it is still an open problem. However, unique solvability of the inverse scattering problem with an infinite number of linearly independent incident fields is guaranteed which we refer to [8]. One of the major advantages of Newton's method is that it can handle the inverse problem with just one incident wave. Based on the Newton's method, our method should inherit this advantage. This leads to the setting of our inverse problem with just one incident wave.

Before we start to solve the inverse problem, we first reformulate it into an equivalent system of equations which can be treated mathematically. In terms of the unknowns ∂D, φ, we define the following two boundary integral operators:

$$B : C^2(\partial D) \times C^{1,\zeta}(\partial D) \to C^{0,\zeta}(\partial D)$$

and

$$F_\infty : C^2(\partial D) \times C^{1,\zeta}(\partial D) \to C^\infty(\Omega)$$

by

$$B(\partial D, \varphi) := \mathrm{i}\frac{\partial}{\partial\vartheta} S \frac{\partial\varphi}{\partial\vartheta} + \mathrm{i}k^2\langle v, S\varphi v\rangle + K'\varphi - \varphi + \mathrm{i}k\lambda(S + \mathrm{i}K + \mathrm{i}I)\varphi \tag{12}$$

and

$$F_\infty(\partial D, \varphi)(\hat{x}) := \int_{\partial D} (\xi_1 + \mathrm{i}\xi_2\langle v(y), \hat{x}\rangle)\mathrm{e}^{-\mathrm{i}k\langle\hat{x},y\rangle}\varphi(y)\,\mathrm{d}s(y) \tag{13}$$

Note that the operator B is just the boundary operator which maps the unknowns to the impedance boundary data. The operator F_∞ is just the far field operator which is similar to that defined in the literatures but with two unknowns.

After defining the operators, let us consider the following system of operator equations:

$$\begin{cases} B(\partial D, \varphi) = -\frac{\partial u^{\mathrm{i}}}{\partial v}\big|_{\partial D} \\ F_\infty(\partial D, \varphi) = u_\infty \end{cases} \tag{14}$$

If ∂D solves the inverse scattering problem, then it follows directly from the solution theory of the direct problem that system (14) is satisfied. Conversely, assume that the pair $(\partial D, \varphi)$ solves (14). Then the first part of (14) ensures that the total field u defined in Theorem 1 satisfies the impedance boundary condition on ∂D. The second equation in (14) then avouches the correct far field pattern for the scattered field u^{s}. We have thus the following main theorem.

Theorem 2. *∂D is the solution of the inverse problem if and only if ∂D, φ solve the system of nonlinear integral* Eq. (14).

4. Numerical treatment

To simplify further discussions, let's first parameterize the boundary of the unknown scatterer with a two times continuously differentiable injective function γ defined on $[0, 2\pi]$ with $\|\gamma'\| \neq 0$. With the substitution $\psi(t) = \varphi(\gamma(t))$, the system (14) takes the following parameterized form:

$$\begin{cases} A_1(\gamma, \psi') + A_2(\gamma, \psi) - k\lambda\psi - \psi = a(\gamma) \\ A_\infty(\gamma, \psi) = u_\infty \end{cases} \tag{15}$$

where the integral operators $A_\omega, \omega = 1, 2, \infty$ are defined by

$$A_\omega(\gamma, \chi) := \int_0^{2\pi} K_\omega(\tau, \sigma)\chi(\sigma)\,\mathrm{d}\sigma \tag{16}$$

with corresponding kernels K_ω which can be easily deduced from (14). Since the system (14) is nonlinear, a linear model can be obtained by the Newton's method, i.e., we have to solve the following linearized system:

$$\begin{cases} A_1(\gamma,\psi') + A_1(\gamma,\chi') + A_1'(\gamma,\psi';q) + A_2(\gamma,\psi) + A_2(\gamma,\chi) + A_2'(\gamma,\psi;q) \\ -k\lambda(\psi+\chi) - (\psi+\chi) = a(\gamma) + a'(\gamma;q) \\ A_\infty(\gamma,\psi) + A_\infty(\gamma,\chi) + A_\infty'(\gamma,\psi;q) = u_\infty \end{cases} \tag{17}$$

where the Fréchet derivatives A'_ω of A_ω, $\omega = 1, 2, \infty$ can be obtained by the Fréchet derivatives of the corresponding kernels as in [11]. Numerically this system is solved iteratively: with a current approximation (ψ,γ), the system (17) has to be solved for the pair (χ, q). The updated data are then given by $\psi + \chi$ for the density function of the integral operators and $\gamma + q$ for the boundary of the unknown scatterer. For brevity, we rewrite the system (17) in the operator form:

$$\mathcal{A}(\chi, q) = \mathcal{Y} \tag{18}$$

Since this linear equation of the first kind is still ill-posed, we have to incorporate some regularization scheme. Instead of solving (18) directly, we solve the following regularized equation:

$$\left(\begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} + \mathcal{A}^*\mathcal{A}\right)\begin{bmatrix} \chi \\ q \end{bmatrix} = \mathcal{A}^*\mathcal{Y} \tag{19}$$

with two to be determined regularization parameters α and β. Numerically we solve this equation by Nyström method which will be described in details in the next section. At this place we want to point out that from the numerical point of view our schema is very attractive. Our method distinguishes from the traditional Newton-type methods in the way that instead of solving a direct problem at each step, the Fréchet derivatives of the integral operators are computed directly by solving the system of integral equation (17). This linear system can be easily solved for example by Gaussian elimination.

5. Numerical results

In this final section of the paper we will demonstrate the efficiency of our method by some examples. In all our examples, we use just one incident wave to reconstruct the scatterer (see also [6] for a sound-soft obstacle). The direct problem is solved once for each example to provide the synthetic far field pattern as the input data for our inverse algorithm. The forward problem can be solved by simply applying the Nyström method to (11). In order to avoid committing an inverse crime, the number of collocation points used in the inverse solver is chosen to be different from that of the forward solver. For the finite dimensional approximate solution, we define the following space of trigonometric functions:

$$T_n(\mathbb{K}) = \left\{ \psi : [0, 2\pi] \to \mathbb{K} \,\middle|\, \psi(\sigma) = \sum_{k=0}^{n} a_k \cos k\sigma + \sum_{k=1}^{n-1} b_k \sin k\sigma,\ a_k, b_k \in \mathbb{K} \right\} \quad \text{for } \mathbb{K} = \mathbb{R}, \mathbb{C}.$$

The solution space for the density function of the integral operators is taken to be the space $T_n(\mathbb{C})$. For the parameterization of the unknown boundary, we choose the space

$$V_m = T_m(\mathbb{R}) \times T_m(\mathbb{R})$$

Generally speaking, for the sake of solvability of the system (19), the parameter m must satisfy the condition $m \leqslant n/2$. In the actual computation, m is much smaller than n.

In all our examples, the incident direction d is taken to be $\left(1/\sqrt{2}, 1/\sqrt{2}\right)^t$. The quadrature points for the direct solver is chosen to be 64 equidistant points in $[0, 2\pi]$. The far field pattern at 40 different directions for the same incident wave are calculated, i.e., $u_\infty(\hat{x}_j, d), j = 1, 2, \ldots, 40$. For the inverse problems, the number of collocation points is taken to be $n = 32$. As the starting curve, that is, the initial guess for the regularized Newton's method, we simply take circles with different radius ρ. The stopping criterion for the iterative scheme is given by the relative error

$$\frac{\|\gamma_{N+1} - \gamma_N\|_2}{\|\gamma_N\|_2} < \epsilon$$

which is taken to be 10^{-5} in all examples. The regularization parameters α for the density ψ and β for the boundary of the scatterer γ are determined by trial and error. Note also that during the reconstruction, all

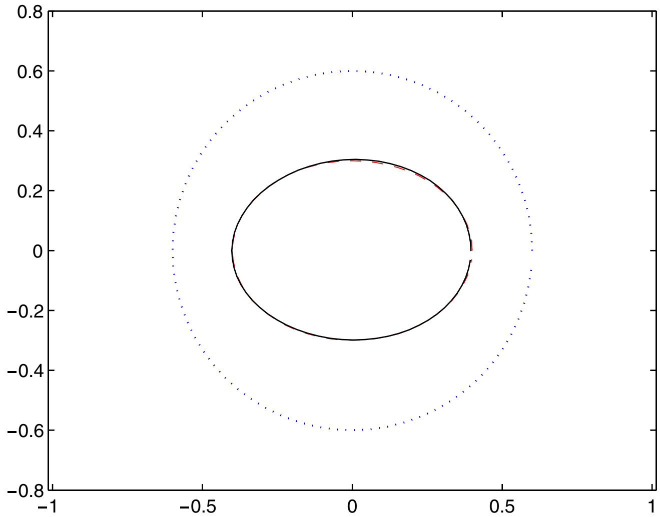

Fig. 1. $\lambda = 1$, k = 1, $\alpha = 1.95\mathrm{E}^{-3}$, $\beta = 9.77\mathrm{E}^{-4}$, $N = 8$.

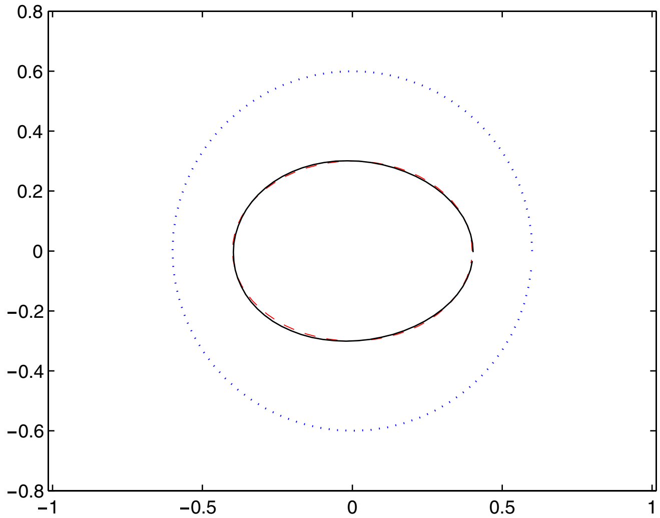

Fig. 2. $\lambda = \sin t$, $k = 1$, $\alpha = 1.53\mathrm{E}^{-5}$, $\beta = 2.98\mathrm{E}^{-8}$, $N = 10$.

parameters (m, α, β) are held fixed. In all our figures below, the dotted line (blue)[1] represents the initial guess. We denote by the dashed line (red) the true solution and by the solid line (black) the reconstruction.

Example 1. For the first example, we take the ellipse $\Gamma = (0.4\cos t, 0.3\sin t)$, $t \in [0, 2\pi]$. As the initial guess for the Newton's method, we choose the circle with radius $\rho = 0.6$. For the case where the impedance $\lambda(t) = 1$, the numerical computation stops at $N = 8$ with the regularization parameters $\alpha = 1.95\mathrm{E}^{-3}$, $\beta = 9.77\mathrm{E}^{-4}$ (Fig. 1). For the case where the impedance $\lambda(t) = \sin t$, our algorithm stops at $N = 10$ with the regularization parameters $\alpha = 1.53\mathrm{E}^{-5}$, $\beta = 2.98\mathrm{E}^{-8}$ (Fig. 2). In this example we see that the convergence of our method is very fast. Only few iterations are needed to achieve the desired result both for the case where λ is a constant and the case of a function. In every iteration step, we only have to solve the regularized Eq. (19) once for the pair (χ, q).

[1] For interpretation of color in Figs. 1–6, the reader is referred to the web version of this article.

Example 2. For the second example, we take a peanut parameterized by

$$\Gamma = \left(\sqrt{\cos^2 t + 0.25\sin^2 t} \times \cos t, \sqrt{\cos^2 t + 0.25\sin^2 t} \times \sin t\right), \quad t \in [0, 2\pi].$$

We start with the circle of radius $\rho = 1.2$. For the case where the impedance $\lambda(t) = 1$, the numerical computation stops at $N = 20$ with the regularization parameters $\alpha = 5.2\mathrm{E}^{-12}$, $\beta = 3\mathrm{E}^{-13}$ (Fig. 3). For the case where the impedance $\lambda(t) = \sin t$, our algorithm stops at $N = 29$ with the regularization parameters $\alpha = 7.28\mathrm{E}^{-12}$, $\beta = 7.28\mathrm{E}^{-12}$ (Fig. 4). Again we see that the convergence of our method is fast, even for this non-convex scatterer.

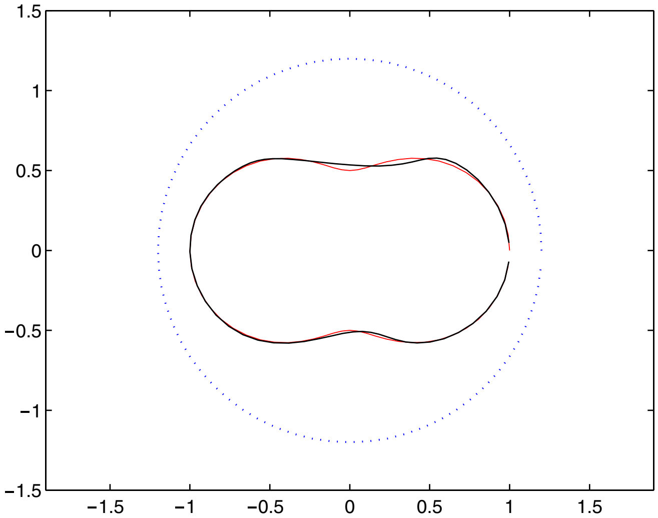

Fig. 3. $\lambda = 1$, $k = 1$, $\alpha = 5.2\mathrm{E}^{-12}$, $\beta = 3\mathrm{E}^{-13}$, $N = 20$.

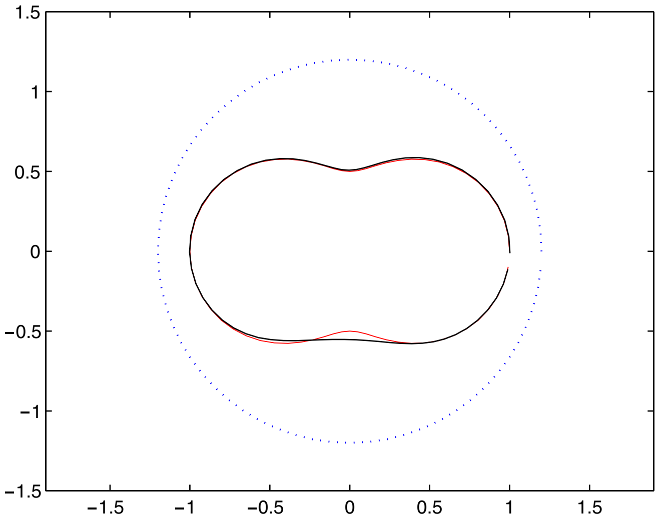

Fig. 4. $\lambda = \sin t$, $k = 1$, $\alpha = 7.28\mathrm{E}^{-12}$, $\beta = 7.28\mathrm{E}^{-12}$, $N = 29$.

Example 3. In the last example, we consider the reconstruction with far field data containinated by 3% Gaussian noise. For this purpose we take again the ellipse $\Gamma = (0.4\ \cos t, 0.3\ \sin t)$, $t \in [0, 2\pi]$. As in the first example, the circle with radius $\rho = 0.6$ serves as the starting curve. For the constant impedance $\lambda = 1$, the numerical computation achieves the desired result at $N = 19$ with the regularization parameters $\alpha = 1.22\text{E}^{-4}$, $\beta = 6.1\text{E}^{-5}$ (Fig. 5). For the impedance function $\lambda(t) = \sin t$, the required accuracy is obtained at $N = 19$ with the regularization parameters $\alpha = 9.54\text{E}^{-7}$, $\beta = 1.22\text{E}^{-4}$ (Fig. 6). This example demonstrates the robustness of the numerical method. The algorithm converges quickly also for erroneous data.

From the examples above, we conclude that our method is fast, accurate and stable. From the schema we also see that the method is conceptually very simple and easy to implement. Finally we want to point out that the choice of the regularization parameters α, β are by try and error. Hence it is not guaranteed that they are the best possible. Furthermore, the parameters are very small as compared to those in [10]. This reflects the fact that the impedance problem is more subtle than the Dirichlet or the Neumann problem.

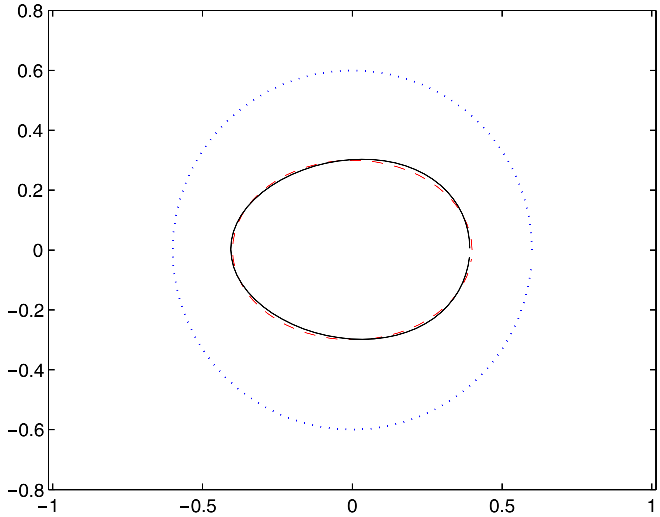

Fig. 5. $\lambda = 1$, $k = 1$, $\alpha = 1.22\text{E}^{-4}$, $\beta = 6.1\text{E}^{-5}$, $N = 19$.

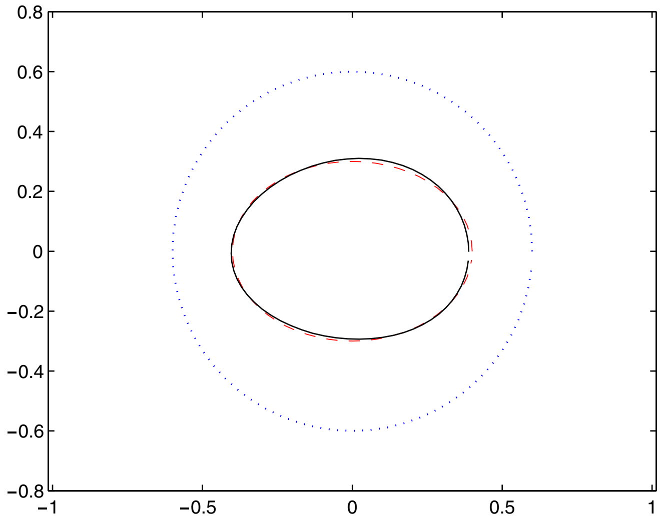

Fig. 6. $\lambda = \sin t$, $k = 1$, $\alpha = 9.54\text{E}^{-7}$, $\beta = 1.22\text{E}^{-4}$, $N = 19$.

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